

QUANTUM DRINFELD ORBIFOLD ALGEBRAS

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ABSTRACT. Quantum Drinfeld orbifold algebras are the generalizations of Drinfeld orbifold algebras, which are obtained by replacing polynomial rings by quantum polynomial rings. In [6], the authors give necessary and sufficient conditions on a defining parameters to obtain Drinfeld orbifold algebras. In this article we generalize their result. It also simultaneously generalizes the result of [4] about quantum Drinfeld Hecke algebras.

1. INTRODUCTION

Drinfeld orbifold algebras arise in different settings, for example, as Lusztig's graded affine Hecke algebras, symplectic reflection algebras, and rational Cherednik algebras. These algebras are deformations of skew group algebra generated by a finite group G which acts on a polynomial ring over some vector space V .

In [6], Shepler and Witherspoon considered the quotient of the skew group algebra $T(V)\#G$ (defined below), where $T(V)$ is the tensor algebra and G is a finite group acting by linear transformations on a finite dimensional vector space V over a field \mathbb{k} . They defined the resulting algebra to be a Drinfeld orbifold algebra if it satisfies the Poincaré-Birkhoff-Witt (PBW) property (defined below). The quotient is a deformation of skew group algebra $S\#G$ (defined below) where S is the symmetric algebra with the induced action of G by automorphisms. These kind of algebras were studied by Halbout, Oudom, and Tang [3], where a finite group G acts faithfully on real vector space V . In this article we replace symmetric algebra by quantum symmetric algebra and express the conditions on algebra parameters in algebraic format to satisfy PBW property. For examples, we refer reader to [4], [5] and [6].

Let \mathbb{k} be a field, and let V be a finite-dimensional vector space over \mathbb{k} . Let v_1, v_2, \dots, v_n be a basis for V , and let $\mathbf{q} := (q_{ij})_{1 \leq i, j \leq n}$ be a tuple of nonzero scalars for which $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ for all i, j .

Let $S_{\mathbf{q}}(V)$ denote the **quantum symmetric algebra**:

$$S_{\mathbf{q}}(V) := \mathbb{k}\langle v_1, \dots, v_n \mid v_i v_j = q_{ij} v_j v_i \text{ for all } 1 \leq i, j \leq n \rangle.$$

Let G be a finite group acting linearly on V , and that there is an induced action on $S_{\mathbf{q}}(V)$ by algebra automorphisms. Then we may form the skew group algebra $S_{\mathbf{q}}(V)\#G$: Letting $A = S_{\mathbf{q}}(V)$, additively $A\#G$ is the free left A -module with basis G . We write $A\#G = \bigoplus_{g \in G} A_g$, where $A_g = \{a\#g \mid a \in A\}$, that is for each $a \in A$ and $g \in G$ we denote $a\#g \in A_g$ the a -multiple of g . Multiplication on $A\#G$ is determined by

$$(a\#g)(b\#h) := a(gb)\#gh$$

for all $a, b \in A$ and $g, h \in G$. Similarly we define $T(V)\#G$ where G acts by automorphisms on the tensor algebra $T(V)$.

Let $\kappa : V \times V \rightarrow (\mathbb{k} \oplus V) \otimes \mathbb{k}G$ be a bilinear map for which $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ for all $1 \leq i, j \leq n$, and $\{t_g \mid g \in G\}$ be a basis of the group algebra $\mathbb{k}G$. Define

$$\mathcal{H}_{\mathbf{q}, \kappa} := T(V) \# G / (v_i v_j - q_{ij} v_j v_i - \kappa(v_i, v_j) \mid 1 \leq i, j \leq n),$$

the quotient of the skew group algebra $T(V) \# G$ by the ideal generated by all elements of the form specified. Giving each v_i degree 1 and each group element g degree 0, $\mathcal{H}_{\mathbf{q}, \kappa}$ is a filtered algebra. We say that $\mathcal{H}_{\mathbf{q}, \kappa}$ satisfies the **PBW condition** if one of the following equivalent conditions holds:

- (1) The associated graded algebra of $\mathcal{H}_{\mathbf{q}, \kappa}$ is isomorphic to $S_{\mathbf{q}}(V) \# G$, as graded algebras.
- (2) The set $\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \geq 0, g \in G\}$ is a \mathbb{k} -basis for $\mathcal{H}_{\mathbf{q}, \kappa}$.

We will call $\mathcal{H}_{\mathbf{q}, \kappa}$ a **quantum Drinfeld orbifold algebra** if it satisfies the PBW condition. In the case when all $q_{ij} = 1$, these are the Drinfeld orbifold algebras studied by [?].

2. NECESSARY AND SUFFICIENT CONDITIONS

For each $g \in G$, let $\kappa_g : V \times V \rightarrow \mathbb{k} \oplus V$ be the function determined by the condition

$$\kappa(v, w) = \sum_{g \in G} \kappa_g(v, w) t_g \quad \text{for all } v, w \in V.$$

Furthermore, let $\kappa_g^C : V \times V \rightarrow \mathbb{k}$ and $\kappa_g^L : V \times V \rightarrow V$ be the functions determined by the condition

$$\kappa_g(v, w) = \kappa_g^C(v, w) + \kappa_g^L(v, w) \quad \text{for all } v, w \in V$$

where κ_g^C and κ_g^L are constant and linear parts of κ_g . The condition $\kappa(v_i, v_j) = -q_{ij}\kappa(v_j, v_i)$ implies that $\kappa_g^C(v_i, v_j) = -q_{ij}\kappa_g^C(v_j, v_i)$ and $\kappa_g^L(v_i, v_j) = -q_{ij}\kappa_g^L(v_j, v_i)$ for each $g \in G$.

For each group element $g \in G$, let g_i^j denote the scalar determined by the equation

$${}^g v_j = \sum_{i=1}^n g_i^j v_i.$$

Define the **quantum (i, j, k, l) -minor determinant** of g as

$$\det_{ijkl}(g) := g_l^j g_k^i - q_{ji} g_l^i g_k^j.$$

The following lemma will be used in the proof of the Theorem 2.2 below.

Lemma 2.1. *Let $g \in G$. We have:*

- (i) $q_{lk} \det_{ijkl}(g) = -\det_{ijlk}(g)$ for all i, j, k, l .
- (ii) For each i, j , if $q_{ij} \neq 1$, then $g_k^i g_k^j = 0$ for all k .

Proof. Refer Lemma 3.2 and Corollary 3.6 in [4]. □

In the proof of the following theorem, we will assume that the reader is familiar with G. Bergman's 1978 paper on the Diamond Lemma [1]. We can also approach the following proof using techniques of Braverman and Gaitsgory [2].

Theorem 2.2. *The algebra $\mathcal{H}_{\mathbf{q}, \kappa}$ is a quantum Drinfeld orbifold algebra if and only if the following conditions hold:*

$$\begin{aligned}
(1) & \text{ For all } g, h \in G \text{ and } 1 \leq i < j \leq n, \\
\kappa_g^C(v_j, v_i) &= \sum_{k < l} \det_{ijkl}(h) \kappa_{hgh^{-1}}^C(v_l, v_k) \quad \text{and} \quad {}^h(\kappa_g^L(v_j, v_i)) = \sum_{k < l} \det_{ijkl}(h) \kappa_{hgh^{-1}}^L(v_l, v_k), \\
(2) & \kappa_g^L(v_k, v_j)(q_{ji}q_{ki}v_i - {}^g v_i) + \kappa_g^L(v_i, v_k)(q_{kj}q_{ki}v_j - q_{ji}q_{ki}{}^g v_j) + \kappa_g^L(v_j, v_i)(v_k - q_{kj}q_{ki}{}^g v_k) = 0, \\
(3) & \sum_{h \in G} \{ \kappa_{gh^{-1}}^L(\kappa_h^L(v_k, v_j), (q_{ji}q_{ki}q_{im}v_i + {}^h v_i)) + \kappa_{gh^{-1}}^L(\kappa_h^L(v_i, v_k), (q_{kj}q_{ki}q_{jm}v_j + q_{ji}q_{ki}{}^h v_j)) + \\
& \quad \kappa_{gh^{-1}}^L(\kappa_h^L(v_j, v_i), q_{km}v_k + q_{kj}q_{ki}{}^h v_k)) \} \\
&= 2\{ \kappa_g^C(v_k, v_j)(q_{ji}q_{ki}v_i - {}^g v_i) + \kappa_g^C(v_i, v_k)(q_{kj}q_{ki}v_j - q_{ji}q_{ki}{}^g v_j) + \kappa_g^C(v_j, v_i)(v_k - q_{kj}q_{ki}{}^g v_k) \}, \\
(4) & \sum_{h \in G} \{ \kappa_{gh^{-1}}^C(\kappa_h^L(v_k, v_j), (q_{ji}q_{ki}q_{im}v_i + {}^h v_i)) + \kappa_{gh^{-1}}^C(\kappa_h^L(v_i, v_k), (q_{kj}q_{ki}q_{jm}v_j + q_{ji}q_{ki}{}^h v_j)) + \\
& \quad \kappa_{gh^{-1}}^C(\kappa_h^L(v_j, v_i), q_{km}v_k + q_{kj}q_{ki}{}^h v_k)) \} = 0
\end{aligned}$$

where $m \notin \{i, j, k\}$.

Proof. We begin by expressing the algebra $\mathcal{H}_{\mathbf{q}, \kappa}$ as a quotient of a free associative \mathbb{k} -algebra. Let $X = \{v_1, v_2, \dots, v_n\} \cup \{t_g \mid g \in G\}$, and let $\mathbb{k}\langle X \rangle$ be the free associative \mathbb{k} -algebra generated by X . Consider the reduction system

$$S = \{(t_g v_i, {}^g v_i t_g), (t_g t_h, t_{gh}), (v_j v_i, q_{ji} v_i v_j + \kappa(v_j, v_i)) \mid g, h \in G, 1 \leq i < j \leq n\}$$

for $\mathbb{k}\langle X \rangle$. Let I be the ideal of $\mathbb{k}\langle X \rangle$ generated by the following elements:

$$t_g v_i - {}^g v_i t_g, \quad t_g t_h - t_{gh}, \quad v_j v_i - q_{ji} v_i v_j - \kappa(v_j, v_i), \quad g, h \in G, 1 \leq i < j \leq n.$$

In what follows, we will use the Diamond Lemma [2] to show that the set

$$\{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \geq 0, g \in G\}$$

is a \mathbb{k} -basis for $\mathbb{k}\langle X \rangle / I$ if and only if the two conditions in the statement of the theorem hold.

Define a partial order \leq on the free semigroup $\langle X \rangle$ as follows: First, we declare that $v_1 < v_2 < \cdots < v_n < g$ for all $g \in G$, and then we set $A < B$ if

- (i) A is of smaller length than B , or
- (ii) A and B have the same length but A is less than B relative to the lexicographic order.

Then \leq is a semigroup partial order on $\langle X \rangle$, compatible with the reduction system S , and having the descending chain condition. Thus, the hypothesis of the Diamond Lemma holds.

Observe that the set $\langle X \rangle_{\text{irr}}$ of irreducible elements of $\langle X \rangle$ is precisely the alleged \mathbb{k} -basis for $\mathbb{k}\langle X \rangle / I$. That is,

$$\langle X \rangle_{\text{irr}} = \{v_1^{m_1} v_2^{m_2} \cdots v_n^{m_n} t_g \mid m_i \geq 0, g \in G\}.$$

In what follows, we show that all ambiguities of S are resolvable. The theorem will then follow by the Diamond Lemma. There are no inclusion ambiguities, but there do exist overlap ambiguities, and these correspond to the monomials

$$t_g t_h t_k, \quad t_g t_h v_i, \quad t_h v_j v_i, \quad v_k v_j v_i, \quad \text{where } 1 \leq i < j < k \leq n, g, h \in G.$$

Associativity of the multiplication in G implies that the ambiguity corresponding to the monomial $t_g t_h t_k$ is resolvable. The equality ${}^{gh}v_i = {}^g({}^h v_i)$ implies that the ambiguity corresponding to the monomial $t_g t_h v_i$ is resolvable. Next, we show that the ambiguity corresponding to the monomial $t_h v_j v_i$ is resolvable if and only if condition (1) in the statement of the theorem holds. Below, we use the symbol “ \longrightarrow ” to indicate that a reduction has been applied. We have

$$\begin{aligned}
t_h v_j v_i &\longrightarrow q_{ji} t_h v_i v_j + t_h \kappa(v_j, v_i) \\
&\longrightarrow q_{ji} {}^h v_i {}^h v_j t_h + t_h \kappa(v_j, v_i) \\
&= q_{ji} \left(\sum_{l=1}^n h_l^i v_l \right) \left(\sum_{k=1}^n h_k^j v_k \right) t_h + t_h \kappa(v_j, v_i) \\
&= q_{ji} \sum_{l < k} h_l^i h_k^j v_l v_k t_h + q_{ji} \sum_{k < l} h_l^i h_k^j v_l v_k t_h + q_{ji} \sum_{k=1}^n h_k^i h_k^j v_k^2 t_h + t_h \kappa(v_j, v_i) \\
&\longrightarrow q_{ji} \sum_{l < k} h_l^i h_k^j v_l v_k t_h + q_{ji} \sum_{k < l} h_l^i h_k^j q_{lk} v_k v_l t_h + q_{ji} \sum_{k < l} h_l^i h_k^j \kappa(v_l, v_k) t_h + q_{ji} \sum_{k=1}^n h_k^i h_k^j v_k^2 t_h + t_h \kappa(v_j, v_i) \\
&\longrightarrow q_{ji} \sum_{k < l} \left(h_k^i h_l^j + q_{lk} h_l^i h_k^j \right) v_k v_l t_h + q_{ji} \sum_{k=1}^n h_k^i h_k^j v_k^2 t_h + q_{ji} \sum_{g \in G} \left(\sum_{k < l} h_l^i h_k^j \kappa_g(v_l, v_k) \right) t_{gh} + \sum_{g \in G} t_h \kappa_g(v_j, v_i) t_g \\
&\longrightarrow q_{ji} \sum_{k < l} \left(h_k^i h_l^j + q_{lk} h_l^i h_k^j \right) v_k v_l t_h + q_{ji} \sum_{k=1}^n h_k^i h_k^j v_k^2 t_h + q_{ji} \sum_{g \in G} \left(\sum_{k < l} h_l^i h_k^j \kappa_g^C(v_l, v_k) \right) t_{gh} \\
&\quad + q_{ji} \sum_{g \in G} \left(\sum_{k < l} h_l^i h_k^j \kappa_g^L(v_l, v_k) \right) t_{gh} + \sum_{g \in G} \kappa_g^C(v_j, v_i) t_{hg} + \sum_{g \in G} {}^h \left(\kappa_g^L(v_j, v_i) \right) t_{hg} \\
&= q_{ji} \sum_{k < l} \left(h_k^i h_l^j + q_{lk} h_l^i h_k^j \right) v_k v_l t_h + q_{ji} \sum_{k=1}^n h_k^i h_k^j v_k^2 t_h + \sum_{g \in G} \left(q_{ji} \sum_{k < l} h_l^i h_k^j \kappa_{hgh^{-1}}^C(v_l, v_k) + \kappa_g^C(v_j, v_i) \right) t_{hg} \\
&\quad + \sum_{g \in G} \left(q_{ji} \sum_{k < l} h_l^i h_k^j \kappa_{hgh^{-1}}^L(v_l, v_k) + {}^h \left(\kappa_g^L(v_j, v_i) \right) \right) t_{hg}
\end{aligned}$$

and

$$\begin{aligned}
t_h v_j v_i &\longrightarrow {}^h v_j {}^h v_i t_h \\
&= \left(\sum_{l=1}^n h_l^j v_l \right) \left(\sum_{k=1}^n h_k^i v_k \right) t_h \\
&= \sum_{l < k} h_l^j h_k^i v_l v_k t_h + \sum_{k < l} h_l^j h_k^i v_l v_k t_h + \sum_{k=1}^n h_k^j h_k^i v_k^2 t_h \\
&\longrightarrow \sum_{l < k} h_l^j h_k^i v_l v_k t_h + \sum_{k < l} q_{lk} h_l^j h_k^i v_k v_l t_h + \sum_{k < l} h_l^j h_k^i \kappa(v_l, v_k) t_h + \sum_{k=1}^n h_k^j h_k^i v_k^2 t_h \\
&\longrightarrow \sum_{k < l} \left(h_k^j h_l^i + q_{lk} h_l^j h_k^i \right) v_k v_l t_h + \sum_{k=1}^n h_k^j h_k^i v_k^2 t_h + \sum_{g \in G} \left(\sum_{k < l} h_l^j h_k^i \kappa_g(v_l, v_k) \right) t_{gh} \\
&= \sum_{k < l} \left(h_k^j h_l^i + q_{lk} h_l^j h_k^i \right) v_k v_l t_h + \sum_{k=1}^n h_k^j h_k^i v_k^2 t_h + \sum_{g \in G} \left(\sum_{k < l} h_l^j h_k^i \kappa_{hgh^{-1}}^C(v_l, v_k) \right) t_{hg} + \sum_{g \in G} \left(\sum_{k < l} h_l^j h_k^i \kappa_{hgh^{-1}}^L(v_l, v_k) \right) t_{hg}.
\end{aligned}$$

The last expressions in the previous two computations are equal if and only if

- (a) $q_{ji} h_k^i h_l^j + q_{ji} q_{lk} h_l^i h_k^j = h_k^j h_l^i + q_{lk} h_l^j h_k^i$ for all $k < l$,
- (b) $q_{ji} h_k^i h_k^j = h_k^i h_k^j$ for all k , and

(c) for all $g \in G$, we have

$$q_{ji} \sum_{k < l} h_l^i h_k^j \kappa_{hgh^{-1}}^C(v_l, v_k) + \kappa_g^C(v_j, v_i) = \sum_{k < l} h_l^j h_k^i \kappa_{hgh^{-1}}^C(v_l, v_k)$$

and

$$q_{ji} \sum_{k < l} h_l^i h_k^j \kappa_{hgh^{-1}}^L(v_l, v_k) + {}^h(\kappa_g^L(v_j, v_i)) = \sum_{k < l} h_l^j h_k^i \kappa_{hgh^{-1}}^L(v_l, v_k).$$

That (a) and (b) hold follows from part (i) and part (ii) of Lemma 2.1, respectively. The equations in (c) are equivalent to the equations in condition (1) in the statement of the theorem.

Finally, we show that the ambiguity corresponding to the monomial $v_k v_j v_i$ is resolvable if and only if conditions (2)-(4) in the statement of the theorem hold.

$$\begin{aligned} v_k v_j v_i &\longrightarrow q_{ji} v_k v_i v_j + v_k \kappa(v_j, v_i) \\ &= q_{ji} v_k v_i v_j + \sum_{g \in G} \left(v_k \kappa_g^C(v_j, v_i) t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &\longrightarrow q_{ji} (q_{ki} v_i v_k v_j + \kappa(v_k, v_i) v_j) + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &= q_{ji} q_{ki} v_i v_k v_j + q_{ji} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i) t_g v_j + \kappa_g^L(v_k, v_i) t_g v_j \right) + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &\longrightarrow q_{ji} q_{ki} v_i v_k v_j + q_{ji} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i)^g v_j t_g + \kappa_g^L(v_k, v_i)^g v_j t_g \right) + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &\longrightarrow q_{ji} q_{ki} (q_{kj} v_i v_j v_k + v_i \kappa(v_k, v_j)) + q_{ji} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i)^g v_j t_g + \kappa_g^L(v_k, v_i)^g v_j t_g \right) \\ &\quad + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &= q_{ji} q_{ki} q_{kj} v_i v_j v_k + q_{ji} q_{ki} \sum_{g \in G} \left(v_i \kappa_g^C(v_k, v_j) t_g + v_i \kappa_g^L(v_k, v_j) t_g \right) + q_{ji} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i)^g v_j t_g + \kappa_g^L(v_k, v_i)^g v_j t_g \right) \\ &\quad + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \\ &= q_{ji} q_{ki} q_{kj} v_i v_j v_k + q_{ji} q_{ki} \sum_{g \in G} \left(\kappa_g^C(v_k, v_j) v_i t_g + v_i \kappa_g^L(v_k, v_j) t_g \right) + q_{ji} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i)^g v_j t_g + \kappa_g^L(v_k, v_i)^g v_j t_g \right) \\ &\quad + \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) v_k t_g + v_k \kappa_g^L(v_j, v_i) t_g \right) \end{aligned}$$

and

$$\begin{aligned} v_k v_j v_i &\longrightarrow q_{kj} v_j v_k v_i + \kappa(v_k, v_j) v_i \\ &= q_{kj} v_j v_k v_i + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j) t_g v_i + \kappa_g^L(v_k, v_j) t_g v_i \right) \\ &\longrightarrow q_{kj} v_j v_k v_i + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right) \\ &\longrightarrow q_{kj} (q_{ki} v_j v_i v_k + v_j \kappa(v_k, v_i)) + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right) \end{aligned}$$

$$\begin{aligned}
v_k v_j v_i &= q_{kj} q_{ki} v_j v_i v_k + q_{kj} \sum_{g \in G} \left(v_j \kappa_g^C(v_k, v_i) t_g + v_j \kappa_g^L(v_k, v_i) t_g \right) + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right) \\
&\longrightarrow q_{kj} q_{ki} (q_{ji} v_i v_j v_k + \kappa(v_j, v_i) v_k) + q_{kj} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i) v_j t_g + v_j \kappa_g^L(v_k, v_i) t_g \right) \\
&\quad + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right) \\
&= q_{kj} q_{ki} q_{ji} v_i v_j v_k + q_{kj} q_{ki} \sum_{g \in G} \left(\kappa_g^C(v_j, v_i) t_g v_k + \kappa_g^L(v_j, v_i) t_g v_k \right) + q_{kj} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i) v_j t_g + v_j \kappa_g^L(v_k, v_i) t_g \right) \\
&\quad + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right) \\
&\longrightarrow q_{kj} q_{ki} q_{ji} v_i v_j v_k + q_{kj} q_{ki} \sum_{g \in G} \left(\kappa_g^C(v_j, v_i)^g v_k t_g + \kappa_g^L(v_j, v_i)^g v_k t_g \right) + q_{kj} \sum_{g \in G} \left(\kappa_g^C(v_k, v_i) v_j t_g + v_j \kappa_g^L(v_k, v_i) t_g \right) \\
&\quad + \sum_{g \in G} \left(\kappa_g^C(v_k, v_j)^g v_i t_g + \kappa_g^L(v_k, v_j)^g v_i t_g \right).
\end{aligned}$$

The last expressions in the previous two computations are equal if and only if

$$\begin{aligned}
&\sum_{g \in G} \{ \kappa_g^C(v_k, v_j) (q_{ji} q_{ki} v_i - {}^g v_i) + \kappa_g^C(v_i, v_k) (q_{kj} q_{ki} v_j - q_{ji} q_{ki}^g v_j) + \kappa_g^C(v_j, v_i) (v_k - q_{kj} q_{ki}^g v_k) + q_{ji} q_{ki} v_i \kappa_g^L(v_k, v_j) \\
&\quad - \kappa_g^L(v_k, v_j)^g v_i + q_{kj} q_{ki} v_j \kappa_g^L(v_i, v_k) - q_{ji} q_{ki} \kappa_g^L(v_i, v_k)^g v_j + v_k \kappa_g^L(v_j, v_i) - q_{kj} q_{ki} \kappa_g^L(v_j, v_i)^g v_k \} = 0
\end{aligned}$$

Write ${}^g v_a = \sum_l g_l^a v_l$ and $\kappa_g^L(v_a, v_b) = \sum_m C_m^{g,a,b} v_m$ for some g_l^a and $C_m^{g,a,b}$ in \mathbb{k} . Now similar calculations as in proof of Theorem 3.1 [6] using the Diamond lemma will lead us to conditions (2)-(4) in the statement of the theorem. □

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